REPEATED INTEGRALS.

By D. C. Gillespie.

Du Bois Reymond* proved that, when the bounded function $f(x, y)$ has a Riemann double integral over the fundamental rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, then the two repeated integrals exist and are each equal to the double integral. This theorem permits the explanation that the function $f(x, y)$ is an integrable function of x for each y is not implied in the statement that the repeated integral $\int_c^d dy \int_a^b f(x, y)dx$ exists. For obviously the existence of the double integral over the rectangle could not depend on the values of the function along a single line parallel to the x axis. The repeated integral with respect to x with respect to y is said to exist when the repeated upper integral and the repeated lower integral in this same order are equal to each other; *i. e.*,

$$\overline{\int_c^d} dy \overline{\int_a^b} dx = \underline{\int_c^d} dy \underline{\int_a^b} dx.$$

To this theorem of Du Bois Reymond there has now been added a corresponding theorem for Lebesgue integrals. It follows then that if the function $f(x, y)$ possesses a Lebesgue double integral and the repeated Riemann integrals exist they are equal to the Lebesgue double integral and hence to each other.†

W. H. Young, with no hypothesis concerning the existence of the double integral, proved that if the function $f(x, y)$ is an integrable function of x for each y and an integrable function of y for each x, all being Riemann integrals, then both repeated integrals exist.‡ Somewhat later, Lichtenstein by a different method obtained the same result and in addition proved the two repeated integrals were equal.§ In a later paper he extends his results to integrals taken over point sets not comprising all points of the fundamental rectangle.||

* Crelle's Journal, vol. 94 (1883), p. 277.

† Hobson, Theory of Functions of a Real Variable, p. 581.

‡ Monatshefte für Mathematik und Physik, vol. 2 (1910), p. 127. The argument seems to depend on the plane set of points at which $f(x, y)$ is discontinuous with respect to y being measurable.

§ Göttingen Nachrichten, (1910), pp. 468–475.

|| Sitzungsbericht der Berliner Mathematischen Gesellschaft, (1910–11), pp. 55–69.

These interesting papers just miss showing that if the two repeated integrals exist they are equal to each other. A proof of this theorem is given in this note.

If the repeated integral $\int_c^d dy \int_a^b f(x, y)dx$ exists, then the values of y for which

$$\left(\overline{\varphi}(y) \equiv \overline{\int_a^b} f(x, y)dx\right) > \left(\underline{\varphi}(y) \equiv \underline{\int_a^b} f(x, y)dx\right)$$

form a set of measure zero. For the fact that

$$\overline{\varphi}(y) \geqq \underline{\varphi}(y) \text{ and that } \overline{\int_c^d} \overline{\varphi}(y)dy = \underline{\int_c^d} \underline{\varphi}(y)dy$$

shows that

$$\overline{\int_c^d} \overline{\varphi}(y)dy = \underline{\int_c^d} \overline{\varphi}(y)dy,$$

i. e., $\overline{\varphi}(y)$ is an integrable function of y. In the same way one sees that $\underline{\varphi}(y)$ is integrable. Then since $\int_c^d (\overline{\varphi}(y) - \underline{\varphi}(y))dy = 0$ and since the integrand is positive or zero its integral over any sub-interval of (c, d) is also zero. The function $(\overline{\varphi}(y) - \underline{\varphi}(y))$ is therefore an integrable null function* and hence different from zero at a set of measure zero.†

The values of y for which the integral $\int_a^b f(x, y)dx$ exists, being identical with those which satisfy the equation $\overline{\varphi}(y) = \underline{\varphi}(y)$, form a set everywhere dense. Moreover, as both repeated integrals are assumed to exist, the set of values of x, for which

$$\left(\overline{\psi}(x) \equiv \overline{\int_c^d} f(x, y)dy\right) > \left(\underline{\psi}(x) \equiv \underline{\int_c^d} f(x, y)dy\right),$$

also has the measure zero. We shall designate this set by G.

Let us now divide the interval from c to d into n equal parts and write the equation

$$\text{(1)} \qquad \frac{d-c}{n}[f(x, \eta_1) + f(x, \eta_2) + \cdots + f(x, \eta_n)] = \overline{\psi}(x) + R_n(x),$$

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* Hobson, loc. cit., pp. 347 and 348.

† While this condition is necessary for the existence of the repeated $\int_c^d dy \int_a^b f(x, y)dx$ it is of course not sufficient. Example: $f(x, y) = 0$ for x and y both rational, $f(x, y) = 1$ for other values of x and y. The fact that the existence of the repeated integral $\int_c^d dy \int_a^b f(x, y)dx$ does require that the integral $\int_a^b f(x, y)dx$ exist except for a set of values of y of measure zero could, I think, be combined with the results obtained by Lichtenstein to prove our theorem.

where η_i is a point of the ith subdivision for which $f(x, \eta_i)$ is an integrable function of x, and $R_n(x)$ is defined by the equation. The function $R_n(x)$ being the difference between two integrable functions of x is itself integrable, hence integrating both sides of equation (1) with respect to x there results

$$(2) \quad \frac{d-c}{n}[\varphi(\eta_1) + \varphi(\eta_2) + \cdots + \varphi(\eta_n)] = \int_a^b \bar{\psi}(x)dx + \int_a^b R_n(x)dx.$$

The set $G_{n,\epsilon}$ is defined to consist of those values of x which satisfy the inequality $|R_n(x)| > \epsilon$, where ϵ is a positive number. The function $R_n(x)$ is integrable; $|R_n(x)|$ is therefore also integrable and $G_{n,\epsilon}$ is measurable. Suppose n is allowed to vary, becoming infinite, there is then defined an infinite sequence of sets $G_{1,\epsilon}$, $G_{2,\epsilon}$, $G_{3,\epsilon}$ $\cdots$ etc., of measure $m(G_{1,\epsilon})$, $m(G_{2,\epsilon})$, $m(G_{3,\epsilon})$ $\cdots$ etc. respectively. Now the $\operatorname{limit}_{n=\infty} m(G_{n,\epsilon}) = 0$. For if this were not true, an infinite number of these sets would each have measure greater than some positive number c. This, in turn, would require that there exist a set of measure greater than or equal to c each point of which belongs to an infinite number of the defined sets.* This is impossible, since only points of G can belong to an infinite number of the sets.

Both $\bar{\psi}(x)$ and $\frac{d-c}{n}[f(x, \eta_1) + f(x, \eta_2) + \cdots + f(x, \eta_n)]$ are less than or at most equal in absolute value to $u(d-c)$, where u is the least upper bound of $|f(x, y)|$ in the region $a \leqq x \leqq b$, $c \leqq y \leqq d$. It follows from equation (1) that $|R_n(x)| \leqq 2u(d-c)$ and as a consequence $\int_a^b |R_n(x)|dx \leqq m(G_{n,\epsilon})2u(d-c) + (b-a)\epsilon$. Then, finally, since $m(G_{n,\epsilon})$ approaches zero as n becomes infinite, $\operatorname{limit}_{n=\infty} \int_a^b R_n(x)dx = 0$. From equation (2) we obtain

$$\operatorname{limit}_{n=\infty} \frac{d-c}{n}[\varphi(\eta_1) + \varphi(\eta_2) + \cdots + \varphi(\eta_n)] = \int_a^b \bar{\psi}(x)dx + \operatorname{limit}_{n=\infty} \int_a^b R_n(x)dx,$$

or

$$\int_c^d dy \int_a^b f(x, y)dx = \int_a^b dx \int_c^d f(x, y)dy.$$

This theorem may now be extended to a certain class of unbounded functions.

We assume:

1°. $f(x, y) \geqq 0$;†

* Hobson, loc. cit., p. 120, § 93.

† Instead of 1° it is sufficient to assume that the function $f(x, y)$ is not both positively and negatively unbounded.

2°. All integrals are proper integrals, i. e., $f(x, y)$ is a bounded function of y for each x and a bounded function of x for each y; moreover the functions $\overline{\varphi}(y)$ and $\overline{\psi}(x)$ are bounded;

3°. The two repeated integrals exist.

The argument is a repretition of that for bounded functions up to where the inequality $|R_n(x)| \leqq 2u(d - c)$ is obtained. This inequality is meaningless when $f(x, y)$ is unbounded and the proof must be concluded without using it.

From 3° it follows that the limit as n becomes infinite of the left side of equation (2) exists. This fact establishes the existence of the

$$\operatorname*{limit}_{n \doteq \infty} \int_a^b R_n(x)dx.$$

The function $f(x, y)$ being positive or zero, $\overline{\psi}(x)$ is also positive or zero. Since then $\overline{\psi}(x)$ is by hypothesis bounded, $R_n(x) \geqq -P$, where P is the least upper bound of $\overline{\psi}(x)$. Let ϵ be any positive number, $G_{n,\,-\epsilon}$ the set of values of x for which $R_n(x) \leqq -\epsilon$, and $m(G_{n,\,-\epsilon})$ the measure of this set. Now $\operatorname*{limit}_{n=\infty} m(G_{n,\,-\epsilon}) = 0$; for $G_{n,\,-\epsilon}$ is only a part of the set $|R_n(x)| \geqq \epsilon$ and, as has been shown, the measure of this second set approaches zero as n becomes infinite. It follows now from

$$\int_a^b R_n(x)dx \geqq m(G_{n,\,-\epsilon})(-P) + (b - a)(-\epsilon),$$

that the $\operatorname*{limit}_{n \doteq \infty} \int_a^b R_n(x)dx \geqq 0$. This fact being established the result of passing to the limit in equation (2) is

$$\int_a^b dx \int_c^d f(x, y)dy \leqq \int_c^d dy \int_a^b f(x, y)dx.$$

An interchange of the rôles played by x and y in the argument would yield

$$\int_c^d dy \int_a^b f(x, y)dx \leqq \int_a^b dx \int_c^d f(x, y)dy.$$

The two repeated integrals are therefore equal.

If we retain conditions 2° and 3° but allow the function $f(x, y)$ to be both positively and negatively unbounded the theorem is no longer true.

Example:

$$f(0, 0) = 0, f(x, y) = \frac{2(x^3y - xy^3)}{(x^2 + y^2)^3}$$

for other values of x and y.

$$\int_0^1 f(x, y)dy = \frac{xy^2}{(x^2+y^2)^2}\Big]_0^1 = \frac{x}{(x^2+1)^2}$$

$$\int_0^1 \frac{x}{(x^2+1)^2}dx = -\frac{1}{2(1+x^2)}\Big]_0^1 = \frac{1}{4},$$

whereas

$$\int_0^1 f(x_1 y)dx = -\frac{x^2 y}{(x^2+y^2)^2}\Big]_0^1 = \frac{-y}{(1+y^2)^2}$$

and

$$\int_0^1 \frac{-y}{(1+y^2)^2}dy = \frac{1}{2(1+y^2)}\Big]_0^1 = -\frac{1}{4}.$$

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